Exponential & Logarithmic Applications

Compound Interest

In compound interest formulas, A is the balance, P is the principal, r is the annual interest rate (in decimal form), and t is the time in years.

Formulas:

Compounding n times per Year

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Compounding Continuously

$$A = Pe^{rt}$$

Examples:

1. Finding the Annual Interest Rate:
   An investment of $50,000 is made in an account that compounds interest quarterly. After 4 years, the balance in the account is $71,381.07. What is the annual interest rate for this account?

   Solution:

   Formula: $A = P\left(1 + \frac{r}{n}\right)^{nt}$

   Labels:
   Principal $= P = \$50{,}000$
   Amount $= A = \$71{,}381.07$
   Time $= t = 4$ years
   Number of compoundings per year $= n = 4$
   Annual interest rate $= r$ (percent in decimal form)

   Equation:

   $71{,}381.07 = 50{,}000\left(1 + \frac{r}{4}\right)^{(4)(4)}$

   $\frac{71{,}381.07}{50{,}000} = \left(1 + \frac{r}{4}\right)^{16}$ — Divide both sides by 50,000.

   $1.42762 \approx \left(1 + \frac{r}{4}\right)^{16}$ — Simplify

   $(1.42762)^{1/16} \approx 1 + \frac{r}{4}$ — Raise both sides to $\frac{1}{16}$ power.

   $1.0225 \approx 1 + \frac{r}{4}$ — Simplify

   $0.09 \approx r$ — Subtract 1 and then multiply both sides by 4.

   The annual interest rate is approximately 9%.

2. Doubling Time for Continuous Compounding:
   An investment is made in a trust fund at an annual interest rate of 8.75%, compounded continuously. How long will it take for the investment to double?

   Solution:

   Formula: $A = Pe^{rt}$

   $2P = Pe^{0.0875t}$ — To show the doubling of the investment, let $A = 2P$.

   $2 = e^{0.0875t}$ — Divide both sides by P.

   $\ln 2 = 0.0875t$ — Apply the inverse property (take the natural log of both sides).

   $\frac{\ln 2}{0.0875} = t$ — Divide both sides by 0.0875.

   $7.92 \approx t$ — Simplify & round.

   It will take approximately 7.92 years for the investment to double.

Growth and Decay

The balance in an account earning *continuously* compounded interest is just one example of a quantity that increases over time according to the **exponential growth model,** $y = Ce^{kt}$.

> ▶ **Exponential Growth and Decay**
> The mathematical model for exponential growth or decay is given by
>
> $y = Ce^{kt}$.
>
> For this model, t is the time, C is the original amount of the quantity, and y is the amount after time t. The number k is a constant that is determined by the rate of growth. If $k > 0$, the model represents **exponential growth**, and if $k < 0$, it represents **exponential decay**.

Examples:

3. Population Growth:
   A country's population was 2 million in 1990 and 3 million in 2000. What would you predict the population of the country to be in 2010 using the exponential growth model?

   Solution:

   Formula: $y = Ce^{kt}$

   Let $t = 0$ represent the year 1990, then the original population would be 2 million, $C = 2$, so the formula becomes:
   $$y = 2e^{kt}$$
   In the year 2000, $t = 10$, the population was 3 million, $y = 3$. Substituting these values into the formula allows us to solve for k:

   $3 = 2e^{10k}$ Substitute values for t and y.

   $\frac{3}{2} = e^{10k}$ Divide both sides by 2.

   $\ln\frac{3}{2} = 10k$ Apply the inverse property (take the natural log of both sides).

   $\frac{1}{10}\ln\frac{3}{2} = k$ Divide both sides by 10.

   $0.0405 \approx k$ Simplify.

   Finally, you can use this value of k to conclude that the population in the year 2010, $t = 20$, is given by:
   $$2e^{0.0405(20)} = 4.5 \text{ million}$$

4. Radioactive Decay:
   Radioactive iodine is a by-product of some types of nuclear reactors. Its **half-life** is 60 days. That is, after 60 days, a given amount of radioactive iodine will have decayed to half the original amount. Suppose a nuclear accident occurs and releases 20 grams of radioactive iodine. How long will it take for the radioactive iodine to decay to a level of 1 gram?

   Solution:

   Formula: $y = Ce^{kt}$

   The original amount of radioactive iodine is 20 grams, $C = 20$, so the formula becomes:
   $$y = 20e^{kt}$$

Since the half-life is 60 days, when $t = 60$, $y = 10$. Substituting these values into the formula allows us to solve for k:

$10 = 20e^{60k}$ Substitute values for t and y.

$\frac{1}{2} = e^{60k}$ Divide both sides by 20.

$\ln\frac{1}{2} = 60k$ Apply the inverse property (take the natural log of both sides).

$\frac{1}{60}\ln\frac{1}{2} = k$ Divide both sides by 60.

$-0.01155 \approx k$ Simplify.

Finally, you can use this value of k to find the time when the amount is 1 gram, $y = 1$, as follows:

$1 = 20e^{-0.01155t}$ Substitute values for k and y.

$\frac{1}{20} = e^{-0.01155t}$ Divide both sides by 20.

$\ln\frac{1}{20} = -0.01155t$ Apply the inverse property (take the natural log of both sides).

$\frac{1}{-0.01155}\ln\frac{1}{20} = t$ Divide both sides by -0.0115.

$t \approx 259.4$ days Simplify.

So, 20 grams of radioactive iodine will have decayed to 1 gram after about 259.4 days.

Intensity Models

On the **Richter scale**, the magnitude R of an earthquake can be measured by the **Intensity model:**

$R = \log_{10} I$

where I is the intensity of the shock wave.

Example:

5. Earthquake Intensity:
   In 1906, San Francisco experienced an earthquake that measured 8.6 on the Richter scale. In 1989, another earthquake, which measured 7.7 on the Richter scale, struck the same area. Compare the intensities of these two earthquakes.

   Solution:

   The intensity of the 1906 earthquake is given as follows:

   $8.6 = \log_{10} I$ Given

   $10^{8.6} = I$ Apply the inverse property (write in exponential form).

   The intensity of the 1989 earthquake is given as follows:

   $7.7 = \log_{10} I$ Given

   $10^{7.7} = I$ Apply the inverse property (write in exponential form).

   The ratio of these two intensities is:

   $$\frac{I \text{ for } 1906}{I \text{ for } 1989} = \frac{10^{8.6}}{10^{7.7}}$$
   $$= 10^{8.6-7.7}$$
   $$= 10^{0.9}$$
   $$\approx 7.94$$

   Thus, the 1906 earthquake had an intensity that was about eight times greater than the 1989 earthquake.